Empirical Earth rotation model: a consistent way to evaluate Earth orientation parameters.

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Abstract. It is customary to perform analysis of the Earth's rotation in two steps: first, to present results of estimation of the Earth orientation parameters in the form of time series based on a simplified model of variations of the Earth's rotation for a short period of time, and then to process this time series of adjustments by applying smoothing, re-sampling and other numerical algorithms. Although this approach saves computational time, it suffers from self-inconsistency: total Earth orientation parameters depend on a subjective choice of the apriori Earth orientation model, cross-correlations between points of time series are lost, and results of an operational analysis per se have a limited use for end users. An alternative approach of direct estimation of the coefficients of expansion of Euler angle perturbations into basis functions is developed. These coefficients describe the Earth's rotation over entire period of observations and are evaluated simultaneously with station positions, source coordinates and other parameters in a single LSQ solution. In the framework of this approach considerably larger errors in apriori EOP model are tolerated. This approach gives a significant conceptual simplification of representation of the Earth's rotation.

Keywords. astrometry, reference systems, Earth rotation, VLBI

1 Introduction

The complexity of the traditional formalism for describing the Earth's rotation is frustrating. Even in the community of professional astronomers a complete understanding of the procedure of applying reduction for the Earth's rotation in all details is not common, especially of the last IAU 2000, IAU 2006 recommendations (McCarthy & Petit, 2004). However, reduction for the Earth's rotation is not only a subject of academic interest, but is the area for various important applications. The complexity of the formalism increases the probability of misunderstanding. Misunderstanding significantly increases the risk of an error in a software which implements reductions. In the worst case scenario, for instance, a glitch in a navigation equipment of a passenger aircraft controlled by such a program may result in a wreck and loss of life. That is why it is vitally important to simplify description of the Earth's rotation for end users.

An alternative approach for describing the Earth rotation was proposed by Petrov (2006). It was demonstrated that instead of using a very complex mathematical model for the a priori rotation matrix and time series of the Earth orientation parameters which correct that a priori rotation matrix, it is feasible to represent the Earth's rotation in a form of a sum of a very simple, coarse a priori mathematical model and a set of coefficients of expansion of the perturbational rotation into the Fourier and B-spline bases which are estimated from observations.

In this paper the implications of the alternative approach with respect to the traditional approach, its advantages and limitations, are discussed.

2 Definition of the Earth's rotation

Space geodesy techniques, such as GPS, SLR, DORIS and VLBI allow to measure the time intervals or the differences of time intervals of electromagnetic wave propagation from observed bodies to observing stations. Solutions of differential equations of wave propagation ties position vectors of observed bodies and their time derivatives with position vector of observing bodies and their time derivatives. Therefore, a relative motion of observing stations with respect to observed bodies can be determined from analysis of observations. Since the observing stations are fixed with respect to the Earth's crust, it is convenient to express their positions in a terrestrial coordinate system in which a motion of stations would be small, an order of magnitude of Earth's crust deformation, i.e. 10^{-8}. A time-independent reference position vector in such a coordinate system, $\vec{r}_{_T}$ is determined via the following linear transformation:

$$\vec{r}(t)_{_C} = \widehat{\mathcal{M}}_a(t)\,\vec{r}_{_T} + \vec{q}_e(t) \times \vec{r}_{_T} + \vec{d}_a(t)_T + \vec{d}_e(t)_T \quad (1)$$

here $\vec{r}(t)_{_C}$ is the reference position vector in celestial coordinate system, $\vec{d}_a(t)_T$ and $\vec{d}_e(t)_T$ are the time-dependent a priori and empirical station displacement vectors, $\widehat{\mathcal{M}}_a(t)$ is the a priori rotation matrix, $\vec{q}_e(t)$ is the empirical small vector of perturbational rotation. It should be noted that such a transformation does not determine coordinates of position vectors in the terrestrial coordinate system uniquely. Equation 1 should be accompanied with the vector equation of additional boundary conditions:

$$\sum_i^N \vec{q}_e(t) \times \vec{d}_e(t)_i = \mathrm{const} \qquad (2)$$

where summing is performed over all stations of the network. The rotation matrix $\widehat{\mathcal{M}}_a(t)$ and its perturbation $\vec{q}_e(t)$ can be considered as the matrix of Earth's rotation if a) the relative motion of stations with respect to the local areas of the Earth's crust is negligible; b) the reference positions of observed bodies are at reset in an inertial coordinate system. Thus, the Earth's rotation is defined here as a net rotation of a polyhedron of reference positions of observing stations with respect to a polyhedron of reference positions of observed bodies.

This definition of the Earth's rotation follows the materialistic approach: "the Earth's rotation is that what **is observed**". The currently prevailing paradigm tends to consider this phenomena from the idealistic point of view, which involves considering a relative motion of celestial spheres, one of which represents the ideal Earth and another represents the sky (refer to (Moritz, 1987) more details). That approach operates such notions as big circles, poles, axes, ecliptic, true equator of date etc., and follows the tradition which can be traced back to ancient Greece (Pannekoek, 1989). This concept has a mechanical interpretation of a rolling and sliding Poinsot's cones (Poinsot, 1834). However, this concept is not adequate when one needs to describe the Earth's rotation with an accuracy comparable to the precision of modern observations, i.e. $3\text{–}5 \cdot 10^{-10}$ rad. First, the Earth can not be considered as a rigid body, and a notion of the rotation axis, i.e. a set of points to which a distance from any point that belongs to the body does not change with time, is not applicable any more. Second, the presence of high frequency variations in the Earth's rotation, others than retrograde diurnal terms, makes the Poinsot's interpretation inadequate. Thirdly, the complexity of the mathematical model — the so-call MHB2000 expansion (Mathews et al., 2002) consists of more than one thousand separate motions — makes a meaningful interpretation problematic, similar to the situation of description of the planetary motion in terms of Ptolemy's epicycles.

3 Parameterization of the Earth's rotation

Since both $\widehat{\mathcal{M}}(t)$ and vector $\vec{d}(t)$ are functions of time, i.e. infinite sets of points, their evaluation from a finite set of observations can be performed only in the form of representing them via some functions. The choice of these functions we call a mathematical model. The model depends on a finite set of unknown parameters which are determined from observations.

The fundamental problem is that no observation technique, except the laser gyroscope, is sensitive to the instantaneous Earth's rotation vector or its time derivatives *directly*. The rotation angles can be derived using the least square estimation procedure together with evaluation of other parameters, and it requires the accumulation of the sufficient amount of data in order to separate variables. The estimates of the Earth's rotation angles cannot be sampled too fast. A typical sampling rate of estimates is one day, since this allows compensation of a certain type of systematic errors. In some cases the sampling rate can be reduced to several hours.

3.1 Traditional approach

A careful examination of the traditional approach first proposed by Herring et al. (1986), reveals three mathematical models (McCarthy & Petit, 2004): 1) the a priori model, 2) the estimation model, and 3) the post-processing model. Corrections to rotation angles around coordinate axes $A_i(t)$ are parameterized in the form equivalent to

$$
\begin{array}{rcl}
A_1(t) & = & c(t)\,\cos -\Omega_n t \;+\; s(t)\,\sin -\Omega_n t \;+ \\
 & & b_1(t) \qquad\qquad\; +\; \dot b_1(t) * (t - t_0) \\
A_2(t) & = & c(t)\,\sin -\Omega_n t \;-\; s(t)\,\cos -\Omega_n t \;+ \\
 & & b_2(t) \qquad\qquad\; +\; \dot b_2(t) * (t - t_0) \\
A_3(t) & = & b_3(t) \qquad\qquad\; +\; \dot b_3(t) * (t - t_0)
\end{array}
\qquad (3)
$$

where Ω_n is the Earth's angular velocity.

This estimation model is not adequate for a long period of time. Usually parameters $s, c, b, \dot b$ are determined for a given 24 hour period. Then the time series of s_i, c_i, b_i are filtered and smoothed. The coefficients of the interpolating spline are computed using the smoothed time series of these parameters. This interpolating spline for continuous functions $s(t), c(t), b(t)$ defined at the entire interval of observations forms the post-processing model. The a priori rotation matrix $\widehat{\mathcal{M}}_a(t)$ is represented as a product of 12 elementary rotations around coordinate axes. Some angles of these rotations depend on functions $s(t), c(t), b(t)$ from the post-processing model.

The traditional approach has certain disadvantages. The three mathematical models involved contradict each other. The parameter estimation is not optimal: the raw time series minimizes residuals in the least square sense, but the filtered and smoothed series does not; if it did, no smoothing would have been needed. The estimation model 3 has a very small range of validity. In order to compensate its coarseness, a very refined and sophisticated a priori model is required. It should be accurate at a very high level. Small changes in the a priori model results in changes not only adjustments, but in total Earth orientation parameters due to inadequacy of the estimation model 3. To demonstrate it, a series of VLBI solutions was made. The first reference solution, gsf2006d[1], used as a priori the USNO Final time series of pole coordinates and UT1 with the time span of 1 day[2]. Eight parameters, $s, c, b, \dot b$, were estimated for each 24 hour observing session independently. In the second solution the Gaussian noise with the standard deviation 1 nrad was added to the USNO Finals EOP. In other trial solutions the Gaussian noise was passed through the rectangular low-pass digital filter with the frequency cutoffs which correspond to periods of 3, 7, 10 and 15 days. The noise was re-scaled in order to have the standard deviation of 1 nrad in all cases. The rms of the differences in *totals*, the sum of apriori and adjustments, with respect to the reference solution are presented in table 1.

Table 1. The rms of the differences in total EOP of solutions with a priori time series with added Gaussian noise with respect to the reference solution in nrad. The Gaussian noise was passed though the rectangular filter which cut the frequencies that corresponding periods shorter than a threshold. The rms of the noise was 1 nrad in all cases.

Freq. cutoff	Pole coord.	UT1	Nutation $\Delta\epsilon$
Periods $< 1^d$	0.147	0.138	0.148
Periods $< 3^d$	0.082	0.074	0.084
Periods $< 7^d$	0.048	0.021	0.016
Periods $< 10^d$	0.041	0.021	0.008
Periods $< 15^d$	0.038	0.012	0.004

This factor is one of the reasons of the so-called "analysis noise" — the differences in results of analysis of the same data processed by different analysis centers. In order to reduce the influence of errors in a priori to a negligible level, say 1/2 of the formal uncertainty, which is nowadays is at a level of 0.3 nrad, the errors in the a priori EOPs must be very small, less than 1 nrad.

Another disadvantage of the traditional approach is that results of parameter estimation are not usable *directly* by an end-user: they should be post-processed. This makes it difficult to assess the errors of the interpolated time series of $c(t), s(t), b(t)$. Correlations are lost, contribution of errors of the a priori EOP model is not taken into account. This makes evaluation of statistical hypothesis based on estimates of the Earth orientation parameters problematic.

3.2 Alternative approach: expansion of the perturbational rotation into basis functions

We can overcome the setbacks of the traditional approach if we refine the estimation model and make it valid not only over a 24 hour period, but over the entire time range of observations, i.e. several decades. This can be done in a form of expansion of the vector of perturbational rotation $\vec q_e(t)$ into series over basis functions. The choice of basis functions is determined by the nature of

[1] `http://vlbi.gsfc.nasa.gov/soltions/2006d`
[2] `ftp://maia.usno.navy.mil/ser7/finals.all`

the process under investigation. It is proposed to expand $\vec{q}_e(t)$ into Fourier and B-spline bases in this paper:

$$\vec{q}_e(t) = \begin{pmatrix} \sum\limits_{k=1-m}^{n-1} f_{1k}\, B_k^m(t) \\ + \sum\limits_{j=1}^{N} \left(P_j^c \cos\omega_m\, t \;+\; P_j^s \sin\omega_j\, t \right) \\ +t\; (S^c \cos -\Omega_n\, t \;+\; S^s \sin -\Omega_n\, t) \\ \\ \sum\limits_{k=1-m}^{n-1} f_{2k}\, B_k^m(t) \\ + \sum\limits_{j=1}^{N} \left(P_j^c \sin\omega_j\, t \;-\; P_j^s \cos\omega_j\, t \right) \\ +t\; (S^c \sin -\Omega_n\, t \;-\; S^s \cos -\Omega_n\, t) \\ \\ \sum\limits_{k=1-m}^{n-1} f_{3k}\, B_k^m(t) + \\ \sum\limits_{j=1}^{N} \left(E_j^c \cos\omega_j\, t + E_j^s \sin\omega_j\, t \right) \end{pmatrix} \qquad (4)$$

where $B_k^m(t)$ is the B-spline function (Nürnberger, 1989) of degree m determined at a sequence of knots $t_{1-m}, t_{2-m}, \ldots, t_0, t_1, \ldots t_k$; ω_j are the frequencies of external forces; the coefficients $f_{ik}, P_j^c, P_j^s, S^c, S^s, E_j^c, E_j^s$ are the parameters of the expansion; Ω_n is the nominal frequency of the Earth's rotation $7.292\,115\,146\,707 \cdot 10^{-5}$ rad s^{-1}.

The B-spline basis is adequate for modeling a smooth, slow component in $\vec{q}_e(t)$, the Fourier basis is adequate for modeling harmonic variations, the coefficients of the cross-term, S^c, S^s take into account corrections to the precession rate and the ecliptic obliquity rate.

Although the B-spline and Fourier bases alone are orthogonal, the sum of two bases is in general not orthogonal. The estimation model should be complemented by the orthogonality condition:

$$\begin{array}{l} \int\limits_{t_0}^{t_1} \left(\sum\limits_{k=1-m}^{n-1} f_k\, B_k^m(t) \;\cdot\; \sum\limits_{j}^{N} P_j^c \cos\omega_j\, t \right) dt = 0 \\ \int\limits_{t_0}^{t_1} \left(\sum\limits_{k=1-m}^{n-1} f_k\, B_k^m(t) \;\cdot\; \sum\limits_{j}^{N} P_j^c \sin\omega_j\, t \right) dt = 0 \end{array} \qquad (5)$$

The choice of the degree of B-spline and the time span between knots, the number of constituents and the frequencies of the Fourier basis elements are determined by the targeted accuracy of the estimation model. It was demonstrated by Petrov (2006), that the time span 3 days for $q_1(t), q_2(t)$, 1 day for the $q_3(t)$ component, and 740 harmonic constituents is sufficient for keeping the accuracy of the estimation model below a 10^{-11} rad level. The frequencies of the Fourier constituents are selected in such a way that they correspond to excitation caused by a) tide-generating potential; b) the near diurnal free wobble; c) the atmospheric nutations.

There are several approaches for selecting the frequencies of relevant Fourier constituents. In the first approach a theory of the Earth's nutations is used. Since the spectrum of the tide generating potential consists of the series of sharp peaks, including constituents with the amplitudes of the potential and forced nutations higher than a certain limit is sufficient. The near diurnal free wobble and the atmospheric nutation are band-limited processes, so selecting all the frequencies within those bands with a step reciprocal to the length of observations is sufficient for modeling these components.

Some components of the theoretical spectrum have a frequency separation less than $1/T$, where T is the time interval of observations. In this case strong constraints are imposed to force the ratio of complex amplitudes to be the same as theoretical.

Though exploiting theoretical knowledge about the Earth rotation allows to reduce considerably the number of constituents of the Fourier basis, this raises a certain concern. If the estimation model implicitly incorporates theoretical assumptions, strictly speaking these estimates cannot be used for validation of the theory. Although the ratios of amplitudes between close constituents of EOP variations caused by the tidal potential exerted by external bodies have a sound theoretical basis, we should bear in mind that the ultimate goal of comparison of theoretical predictions with observations is to check validity of assumptions put in the foundation of the theory and to make a judgment whether the model is complete or not. If there are unaccounted additive constituents at these frequencies, for example, caused by the free motion, by the atmospheric or oceanic excitation, the theoretical ratios may not be valid.

Anther way to find the sequence of frequencies where the signal is present is to make a set of trial solutions and to estimate amplitudes of harmonic variations of $\vec{q}_e(t)$ at a set of frequencies sampled with a step reciprocal to the length of observations within the diurnal, semi-diurnal and ter-diurnal bands and discard the constituents where no statistically significant signal was detected. This approach is free from a potential bias of adopted theories.

4 Estimation of the vector of the perturbational rotation from VLBI data

The VLBI dataset from January 1984 through September 2006 was used for validation of the proposed approach. On average, 150 twenty four experiments per year have been observed. The number of participating stations in each individual session varies from 2 to 20, although 4–7 is a typical number. No station participated in all sessions, but every station participated in sessions with many different networks. All networks have common nodes and, therefore, are tied together.

The requirement to the accuracy of the a priori Earth's rotation model is determined by a condition that $||\vec{q}_e||^2$ be less than errors of determination of the perturbational vector $\vec{q}_e$, i.e. 10^{-11} rad. This gives the requirement of the accuracy of the a priori matrix: $3 \cdot 10^{-6}$ rad. This is three orders of magnitude lower than the requirement to accuracy of the a priori rotation matrix in accordance with the traditional approach. We can exploit this advantage of the alternative approach and use the simplest possible model. The following expression for the a priori matrix of the Earth's rotation $\widehat{\mathcal{M}}_a(t)$ according to the Newcomb-Andoyer formalism was used:

$$\begin{aligned}\widehat{\mathcal{M}}_a(t) = & \widehat{\mathcal{R}}_3(\zeta_0) \cdot \widehat{\mathcal{R}}_2(\theta_0) \cdot \widehat{\mathcal{R}}_3(z) \cdot \widehat{\mathcal{R}}_1(-\epsilon_0) \cdot \\ & \widehat{\mathcal{R}}_3(\Delta\psi) \cdot \widehat{\mathcal{R}}_1(\epsilon_0 + \Delta\epsilon) \cdot \widehat{\mathcal{R}}_3(-S)\end{aligned} \tag{6}$$

where $\widehat{\mathcal{R}}_i$ is a rotation matrix around the axis i. For the variables $\zeta_0, \theta_0, z, \epsilon_0, \Delta\psi, \Delta\epsilon_0, S$, the following simplified expression were used:

$$\begin{aligned}
\zeta_0 &= \zeta_{00} + \zeta_{01}\, t + \zeta_{02}\, t^2 \\
\theta_0 &= \theta_{00} + \theta_{01}\, t + \theta_{02}\, t^2 \\
z &= z_0 + z_1\, t + z_2\, t^2 \\
\epsilon_0 &= \epsilon_{00} + \epsilon_{01}\, t + \epsilon_{02}\, t^2 \\
\Delta\psi &= \sum_j^2 p_j \sin(\alpha_j + \beta_j\, t) \\
\Delta\epsilon &= \sum_j^2 e_j \cos(\alpha_j + \beta_j\, t) \\
S &= S_0 + E_0 + \Delta\psi \cos\epsilon_0 \\
&\quad + (\Omega_n + \zeta_{01} + z_1 + E_1)\, t \\
&\quad + (\zeta_{02} + z_2 + E_2)\, t^2 \\
&\quad + \sum_i^2 (E_i^c \cos\gamma_i\, t + E_i^s \sin\gamma_i\, t)
\end{aligned} \tag{7}$$

Here t is TAI time elapsed from 2000, January 01, 12 hours. Some of these parameters were taken from theory, some of them were found with LSQ fit of time series of adjustments of pole coordinates and UT1. Errors of this apriori Earth's rotation model are less than $2 \cdot 10^{-6}$ rad over the period 1984–2006.

Unlike to processing GPS observations, analysis of VLBI observations is done in a so-called global mode: a set of 2,000–20,000 global parameters which are considered common over the entire set of observations, i.e. 22 years; 1,000–50,000 local parameters specific for a given 24 hour session; and over a million of segmented nuisance parameters specific for a 20–60 minute interval are solved in a single least square solution directly.

In the present solution global parameters were stations positions, station velocities, amplitudes of harmonic variation in site positions, coefficients of B-spline modeling non-linear site position variations, source coordinates and proper motions, harmonic variations at 740 frequencies and coefficients of B-spline of the perturbational vector of the Earth's rotation $\vec{q}_e$.

4.1 Summary of the VLBI results

Results of analysis are available on the Web at `http://vlbi.gsfc.nasa.gov/erm`. The weighted root mean square of postfit residuals was the same as in the solution that followed the traditional approach, 21.9 ps.

For comparison with the USNO time series of pole coordinates and UT1-TAI, these time series were transformed to the vectors of perturbational rotation, and the coefficients of the cubic interpolating spline was computed. These coefficients represents the USNO mathematical model of the Earth rotation. The accuracy of the empirical Earth rotation model is higher at the instants of time of middle of observing sessions. The values of vectors of perturbational ro-

Table 2. The first row shows the rms of the differences over the period 1996.0–2006.0 between two models of the Earth's rotation: the empirical model and the USNO model. The second row shows the rms of the differences between solution gsf2006d that follows the traditional approach and the USNO model.

Solution	Pole	UT1	Pole rate	UT1 rate
	10^{-9} rad		10^{-14} rad s^{-1}	
ERM	0.64	0.52	0.99	0.81
gsf2006d	0.56	0.42	1.95	1.52

tation and their time derivatives were computed at these moments of time for both the empirical model from the VLBI solution and from the USNO mathematical model. The rms of these differences are presented in table 2. It should be noted that the GPS estimates of pole coordinate almost entirely dominate the USNO time series. Therefore, the differences between pole coordinates and their rates give us a measure of an agreement between the empirical Earth rotation model from VLBI and the *independent* estimates from GPS.

In order to validate the estimates of the harmonic variations of the perturbational rotation vector, a trial solution following the traditional Earth rotation parameterization and the a priori empirical model of the Earth rotation from the previous solution was run. The rms of adjustments of nutation angles over 1996.0–2006.0 with respect to the apriori MHB2000 expansion (Mathews et al., 2002) was 0.98 nrad and 0.39 nrad with respect to the empirical Earth rotation model.

5 Conclusions

It was demonstrated that the empirical Earth rotation model can be determined directly from observations over a period of 22 years using the least square estimation technique. The advantage of the proposed approach is that a continuous function describing the Earth's orientation is determined in one step without producing intermediate time series. Another advantage of the proposed approach is that a simplified a priori model with only 31 numerical parameters is sufficient, while according to the traditional approach a complicated a priori model of precession, nutation, high frequency harmonic variations of the Earth's rotation and a filtered and smoothed time series of the Earth orientation parameters produced in the previous analysis, in total 46 000 numerical parameters (McCarthy & Petit, 2004), is needed.

The proposed approach is conceptually much more simple than the traditional approach, since it does not operate with idealistic notions that are not observable, such as the non-rotating origin, the equinox, various intermediate poles, axes, etc.

The EOP, the station position and velocities, the source coordinates are produced in a single LSQ adjustment, and therefore, are mutually consistent.

It was demonstrated that the empirical Earth rotation model derived from analysis of VLBI observations gives the differences with respect to the EOP derived from analysis of independent GPS observations at moments of observation at the same level, within 20%, as the differences of the VLBI EOP series produced with the traditional approach. The advantage of the proposed approach is that the estimates of the EOP rates are by a factor of 1.5–2.0 closer to the GPS time series than the VLBI EOP rates estimated following the traditional approach. The rms of estimates of residual nutation angles with respect to the empirical Earth rotation model is 2.6 times smaller than the residuals with respect to the MHB2000 expansion.

References

Herring, T.A., Gwinn, C.R. & Shapiro, I.I., 1986, J. Geophys. Res., 91, 4745

Mathews P.M., Herring, T.A., & Buffett B.A., 2002, J. Geophys. Res., 107, 10.1203/2001JB1000390

McCarthy & Petit(2004), ed., IERS Conventions, 2004, Verlag des Bundesamtes fuer Katrografie und Geodaesie, 127

Moritz H., Earth Rotation: Theory and Observation, 1987, Frederick Ungar, 617

Nürnberger, G., 1989, Approximation by spline functions, Springer-Verlag

Pannekoek, A., A history of astronomy, Dover Publications Inc., 1989, 521 p

Petrov, L., The empirical Earth rotation model from VLBI observations, 2006, A&A, in press `http://arxiv.org/astro-ph/0611781`

Poinsot, L. Outlines of a New Theory of Rotatory Motion, Cambridgem, John Smith., 1834